§1.5 The De Witt-Faddeev-Popov Method

Want to derive path integral formula respecting Lorentz invariance.

Consider

$$\mathcal{I} = \int \left[\prod_{n,x} d\phi_n(x)\right] \mathcal{G}[\phi]\, \mathcal{B}[f(\phi)]\, \mathrm{Det}\, \mathcal{F}[\phi] \qquad (1)$$

where $\phi_n(x)$ are set of gauge and matter fields

$\prod_{n,x} d\phi_n(x)$ is volume element

$\mathcal{G}[\phi]$ is functional satisfying gauge inv. condition

$$\mathcal{G}[\phi_\lambda] \prod_{n,x} d\phi_{\lambda n}(x) = \mathcal{G}[\phi] \prod_{n,x} d\phi_n(x)$$

$\uparrow$ gauge trf. op.

$f_\alpha[\phi; x]$ is "gauge fixing functional"

$\parallel$

$A_{\alpha 3}(x)$ is previous result (not gauge inv.)

$\mathcal{B}[f]$ is functional of general functions $f_\alpha(x)$

$\mathcal{F}$ is matrix

$$\mathcal{F}_{\alpha x, \beta y}[\phi] = \left.\frac{\delta f_\alpha[\phi_\lambda; x]}{\delta \lambda_\beta(y)}\right|_{\lambda=0} \qquad (*)$$

Our previous result

$$\int \left[\prod_{\ell,x} d\psi_\ell(x)\right]\left[\prod_{\alpha,\mu,x} dA_{\alpha\mu}(x)\right] \times \mathcal{O}_A \mathcal{O}_B \cdots \exp\{iI + \varepsilon \text{ terms}\}$$

$$\times \prod_{x,\alpha} \delta(A_{\alpha 3}(x)) \qquad (2)$$

is a special case of (1):

$$f_\alpha[A,\psi;x] = A_{\alpha 3}(x),$$

$$\mathcal{B}[f] = \prod_{x,\alpha} \delta(f_\alpha(x))$$

$$\mathcal{G}[A,\psi] = \exp\{iI + \varepsilon \text{ terms}\}\, \mathcal{O}_A \mathcal{O}_B \cdots,$$

$$\prod_{n,x} d\phi_n(x) = \left[\prod_{\ell,x} d\psi_\ell(x)\right]\left[\prod_{\alpha,\mu,x} dA_\alpha^\mu(x)\right]$$

→ (1) and (2) are equal aside from the factor $\mathrm{Det}\,\mathcal{F}[\phi]$

if $A_\alpha^3(x) = 0$ then gauge trf. gives

$$A_\alpha^3(x) = \partial_3 \lambda_\alpha(x) = \int d^4y\, \lambda_\alpha(y)\, \partial_3 \delta^4(x-y)$$

$$\rightarrow \mathcal{F}_{\alpha x, \beta y}[\phi] = \delta_{\alpha\beta}\, \partial_3 \delta^4(x-y)$$

and $\mathcal{B}$ field-independent!

→ irrelevant for path integral (simple normalization factor)

Eploying the formula (1), we can now freely change the gauge.

<u>Theorem:</u>

Integral (1) is independent of $f_\alpha[\phi;x]$, and depends on the choice of the functional $\mathcal{B}[f]$ only through irrelevant constant factor.

Proof:

replace $\phi \longrightarrow \phi_\Lambda$ (gauge transformed)

with $\Lambda^\alpha(x)$ arbitrary gauge trf. parameter

$$I = \int \underbrace{\left[\prod_{n,x} \phi_{\Lambda n}(x)\right] \mathcal{G}[\phi_\Lambda]}_{\text{gauge inv.}} B[f[\phi_\Lambda]] \operatorname{Det} \mathcal{F}[\phi_\Lambda]$$

$$= \int \left[\prod_{n,x} \phi_n(x)\right] \mathcal{G}[\phi] B[f(\phi_\Lambda)] \operatorname{Det} \mathcal{F}[\phi_\Lambda]$$

Since $\Lambda^\alpha(x)$ was arbitrary, we can integrate over it:

$$I \left[\prod_{\alpha,x} d\Lambda^\alpha(x)\right] \underset{\substack{\uparrow \\ \text{some measure}}}{\rho(\Lambda)} = \int \left[\prod_{n,x} d\phi_n(x)\right] \mathcal{G}[\phi] C[\phi]$$

where $$C[\phi] \equiv \int \left[\prod_{\alpha,x} d\Lambda^\alpha(x)\right] \rho[\Lambda] B\left[f[\phi_\Lambda]\right] \operatorname{Det} \mathcal{F}[\phi_\Lambda]$$

$$\longrightarrow \mathcal{F}_{\alpha x, \beta y}[\phi_\Lambda] = \left.\frac{\delta f_\alpha[(\phi_\Lambda)_\lambda ; x]}{\delta \lambda^\beta(y)}\right|_{\lambda=0}$$

gauge transformations form a group, i.e.

$$(\phi_\Lambda)_\lambda = \phi_{\tilde{\Lambda}(\Lambda,\lambda)}$$

$$\rightarrow \mathcal{F}_{\alpha x, \beta y}[\phi_\Lambda] = \int \mathcal{J}_{\alpha x, \gamma z}[\phi, \Lambda] \, \mathcal{R}^{\gamma z}{}_{\beta y}[\Lambda] \, d^4 z$$

where $$\mathcal{J}_{\alpha x, \gamma z}[\phi, \Lambda] \equiv \left.\frac{\delta f_\alpha[\phi_{\tilde{\Lambda}} ; x]}{\delta \tilde{\Lambda}^\gamma(z)}\right|_{\tilde{\Lambda}=\Lambda} = \frac{\delta f_\alpha[\phi_\Lambda ; x]}{\delta \Lambda^\gamma(z)}$$

and $$\mathcal{R}^{\gamma z}{}_{\beta y}[\Lambda] = \left.\frac{\delta \tilde{\Lambda}^{\gamma}(z; \Lambda, \lambda)}{\delta \lambda^{\beta}(y)}\right|_{\lambda=0}$$

$$\Longrightarrow \quad \text{Det}\, \mathcal{F}[\phi_\Lambda] = \text{Det}\, \mathcal{J}[\phi, \Lambda]\, \text{Det}\, \mathcal{R}[\Lambda]$$

↑ Jacobian of transformation from $\Lambda^\alpha(x)$ to $f_\alpha[\phi_\Lambda; x]$

set $\rho(\Lambda) = \dfrac{1}{\text{Det}\,\mathcal{R}[\Lambda]}$

Then
$$C[\phi] = \int \Big[\prod_{\alpha, x} d\Lambda^\alpha(x)\Big] \text{Det}\, \mathcal{J}[\phi, \Lambda]\, B\big[f[\phi_\Lambda]\big]$$
$$= \int \Big[\prod_{\alpha, x} df_\alpha(x)\Big] B(f) \equiv C.$$

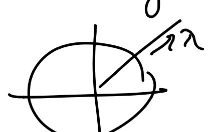
→ independent of ϕ

$$\Longrightarrow \quad I = \frac{C \int \Big[\prod_{n, x} d\phi_n(x)\Big] \mathcal{G}[\phi]}{\int \Big[\prod_{\alpha, x} d\Lambda^\alpha(x)\Big] \rho[\Lambda]} \qquad (3)$$

→ independent of $f_\alpha(\phi; x)$ $\square$

Remark:

Numerator integral is divergent as constant along all gauge orbits



Denominator has the same infinite factor (volume of gauge group × volume of space-time) → both cancel

Using the Theorem, we can rewrite our path integral as follows

$$\langle T\{O_A O_B \cdots\}\rangle_V \sim \int \left[\prod_{\ell,x} d\psi_\ell(x)\right]\left[\prod_{\alpha,\mu,x} dA^\mu{}_\alpha(x)\right]$$

$$\times\, O_A O_B \cdots \exp\{iI + \varepsilon \text{ terms}\}\, B[f[A,\psi]]\, \mathrm{Det}\, \mathcal{F}[A,\psi]$$

for (almost) any choice of f_α and $B[f]$

Now take

$$B[f] = \exp\left(-\frac{i}{2\xi}\int d^4x\, f_\alpha(x) f_\alpha(x)\right)$$

with arbitrary real parameter ξ.

$$\rightarrow \mathcal{L}_{EFF} = \mathcal{L} - \frac{1}{2\xi} f_\alpha f_\alpha$$

choose $f_\alpha = \partial_\mu A^\mu_\alpha$ (Lorentz gauge)

$\rightarrow$ free vector-boson part of effective action:

$$I_{0A} = -\int d^4x \Big[\frac{1}{4}(\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu})(\partial^\mu A_\alpha{}^\nu - \partial^\nu A_\alpha{}^\mu)$$

$$+ \frac{1}{2\xi}(\partial_\mu A_\alpha{}^\mu)(\partial_\nu A_\alpha{}^\nu) + \varepsilon \text{ terms}\Big]$$

$$= -\frac{1}{2}\int d^4x\, \mathcal{D}_{\alpha\mu x,\beta\nu y}\, A_\alpha{}^\mu(x) A_\beta{}^\nu(y),$$

where

$$\mathcal{D}_{\alpha\mu x,\beta\nu y} = \eta_{\mu\nu}\frac{\partial^2}{\partial x^\lambda \partial y_\lambda}\delta^4(x-y)\,\delta_{\alpha\beta}$$

$$-\left(1-\frac{1}{\xi}\right)\frac{\partial^2}{\partial x^\mu \partial y^\nu}\delta^4(x-y)\,\delta_{\alpha\beta} + \varepsilon \text{ terms}$$

$$= (2\pi)^{-4}\,\delta_{\alpha\beta}\int d^4p\left[\eta_{\mu\nu}(p^2 - i\varepsilon) - \left(1 - \frac{1}{\xi}\right)p_\mu p_\nu\right]e^{ip.(x-y)}$$

Taking the inverse gives propagator:

$$\Delta_{\alpha\mu,\beta\nu}(x,y) = (\mathscr{D}^{-1})_{\alpha\mu x,\beta\nu y}$$

$$= (2\pi)^{-4}\delta_{\alpha\beta}\int d^4p\left[\eta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}\right]\frac{e^{ip.(x-y)}}{p^2 - i\varepsilon}$$

→ gives bak Landau gaug for $\xi = 0$
and Feynman gauge for $\xi = 1$
"generalized ξ-gauge"

Feynman rules:
trilinear interaction term in $\mathcal{L}$ gives

$$-\frac{1}{2}C_{\alpha\beta\gamma}(\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu})A_\beta{}^\mu A_\gamma{}^\nu$$

→ contribution to integrand:

$$i(2\pi)^4\,\delta^4(p+q+k)[-iC_{\alpha\beta\gamma}][p_\nu\eta_{\mu\lambda} - p_\lambda\eta_{\mu\nu} + q_\lambda\eta_{\nu\mu} - q_\mu\eta_{\nu\lambda} + k_\mu\eta_{\lambda\nu} - k_\nu\eta_{\lambda\mu}]$$

The A^4 interaction term in $\mathcal{L}$ gives

$$-\frac{1}{4}C_{\varepsilon\alpha\beta}C_{\varepsilon\gamma\delta}A_{\alpha\mu}A_{\beta\nu}A_\gamma{}^\mu A_\delta{}^\nu$$

$$\to i(2\pi)^4\,\delta^4(p+q+k+\ell)\times\Big[-C_{\varepsilon\alpha\beta}C_{\varepsilon\gamma\delta}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})$$

$$-C_{\varepsilon\alpha\gamma}C_{\varepsilon\delta\beta}(\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\sigma\rho})$$

$$-C_{\varepsilon\alpha\delta}C_{\varepsilon\beta\gamma}(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\sigma\nu})\Big].$$

Ghosts:

Up to now we have not yet considered the effect of the factor Det $\mathcal{F}$.

$$\text{Det}\,\mathcal{F} \sim \int \left[\prod_{\alpha, x} d\omega^*_\alpha(x)\right]\left[\prod_{\alpha, x} d\omega_\alpha(x)\right] \exp(i I_{GH})$$

where
$$I_{GH} \equiv \int d^4x\, d^4y\, \omega^*_\alpha(x)\,\omega_\beta(y)\, \mathcal{F}_{\alpha x, \beta y}$$

Here ω^*_α and ω_α are a set of anti-commuting variables.

$$\rightarrow \langle T\{O_A \cdots\}\rangle_V \sim \int \left[\prod_{n, x} d\psi_n(x)\right]\left[\prod_{\alpha,\mu, x} dA_{\alpha\mu}(x)\right]$$
$$\times \left[\prod_{\alpha, x} d\omega_\alpha(x)\, d\omega^*_\alpha(x)\right] \exp\left(i I_{MOD}[\psi, A, \omega, \omega^*]\right) O_A \cdots,$$

where I_{MOD} is
$$I_{MOD} = \int d^4x \left[\mathcal{L} - \frac{1}{2\xi} f_\alpha f_\alpha\right] + I_{GH}$$

ω_α, ω^*_α are called "ghosts" and "antighosts"

Feynman rules:

suppose $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1$

$\mathcal{F}_0$: field independent

$\mathcal{F}_1$: field dependent

$\rightarrow$ ghost propagator:

$$\Delta_{\alpha\beta}(x,y) = -(\tilde{\mathcal{F}}_0^{-1})_{\alpha x,\beta y}$$

vertices:

$$I'_{GH} = \int d^4x\, d^4y\, \omega^*_\alpha(x)\, \omega_\beta(y)\, (\tilde{\mathcal{F}}_1)_{\alpha x,\beta y}$$

Example:

In generalized ξ-gauge we have

$$f_\alpha = \partial_\mu A_\alpha^\mu$$

$$\longrightarrow A^\mu_{\alpha\lambda} = A_\alpha^\mu + \partial^\mu \lambda_\alpha + C_{\alpha\gamma\beta}\lambda_\beta A_\gamma^\mu$$

so that

$$\tilde{\mathcal{F}}_{\alpha x,\beta y} = \left.\frac{\delta \partial_\mu A^\mu_{\alpha\lambda}(x)}{\delta \lambda_\beta(x)}\right|_{\lambda=0}$$

$$= \Box\, \delta^4(x-y) + C_{\alpha\gamma\beta}\frac{\partial}{\partial x^\mu}\left[A_\gamma^\mu(x)\,\delta^4(x-y)\right]$$

$$\rightarrow (\tilde{\mathcal{F}}_0)_{\alpha x,\beta y} = \Box\, \delta^4(x-y)\,\delta_{\alpha\beta}$$

$$(\tilde{\mathcal{F}}_1)_{\alpha x,\beta y} = -C_{\alpha\beta\gamma}\frac{\partial}{\partial x^\mu}\left[A_\gamma^\mu(x)\,\delta^4(x-y)\right]$$

$\rightarrow$ ghost propagator:

$$\Delta_{\alpha\beta}(x,y) = \delta_{\alpha\beta}(2\pi)^{-4}\int d^4p\, (p^2 - i\varepsilon)^{-1} e^{ip\cdot(x-y)}$$

ghost vertex: $i(2\pi)^4\,\delta^4(p+q+k)\times ip_\mu\, C_{\alpha\beta\gamma}$